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On compatibility and improvement of different quantum state assignments

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Abstract
When Alice and Bob have different quantum knowledges or state assignments (density operators) ρ_A and ρ_B, respectively, for one and the same specific individual system, then the problems of compatibility and pooling arise. The so-called first Brun–Finkelstein–Mermin (BFM) condition for compatibility is reobtained in terms of possessed or sharp (i.e., probability one) properties. The second BFM condition is shown to be generally invalid in an infinite-dimensional state space. An argument leading to a procedure of improvement of ρ_A on account of ρ_B and vice versa is presented.

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1. Introduction

To my knowledge, the problem of compatible state assignments originated with Peierls [1]. His necessary conditions were seriously criticized by Fuchs and Mermin [2]. Then Brun, Finkelstein and Mermin (BFM) [3, 4] derived two necessary conditions for the compatibility of different state assignments ρ_A and ρ_B, i.e., for two density operators, describing (being the quantum knowledge about) one and the same system. The two conditions were found to be equivalent in a finite-dimensional state space (to which they confined their discussion).

The first BFM condition reads that the intersection of the supports is at least one dimensional:

$$\dim\{\mathrm{supp}(\rho_A) \cap \mathrm{supp}(\rho_B)\} \geqslant 1. \tag{1}$$

(Support of a density operator is the subspace spanned by the eigensubspaces corresponding to positive eigenvalues.)

The second BFM condition states that there exist pure-state expansions

$$\rho_A = p_A|\phi\rangle\langle\phi| + \sum_{i\geqslant 1} p_{Ai}|\phi_{Ai}\rangle\langle\phi_{Ai}| \qquad \rho_B = p_B|\phi\rangle\langle\phi| + \sum_{i\geqslant 1} p_{Bi}|\phi_{Bi}\rangle\langle\phi_{Bi}| \tag{2}$$

(with all weights non-negative and p_A and p_B both positive) having a common pure state $|\phi\rangle$.

In [5] two approaches to compatibility are discussed. One is shown to be equivalent to the BFM condition. The other leads to a hierarchy of measurement-based compatibility criteria, all inequivalent with the BFM condition.

In [6] a more general approach based on a measure of the compatibility between two state assignments is expounded. This measure is then applied to a procedure of pooling information.

In [7] classical and quantum pooling of information is discussed. The author claims that in the quantum case Alice and Bob must also possess information about how their respective states of knowledge were obtained.

Most likely there are some more or less important contributions that I have unintentionally omitted in this very short review because I am not aware of them.

In [4] a thorough list of further questions is given. Clearly, the problem of compatibility and of pooling of information from different state assignments (for one and the same individual system) is not quite near to its complete solution.

Henceforth the state space is allowed to be finite or countably infinite dimensional.

This paper is organized as follows. In section 2 the first BFM necessary condition is reobtained in a mathematically slightly but physically considerably different way than in the BFM paper [3]. (The approach of section 2 is required for section 4.) In section 3 it is shown that in an infinite-dimensional state space the two BFM conditions are not equivalent, and that only the first one is generally valid. In section 4 a possible improvement of ρ_A on account of ρ_B and vice versa is expounded, and a simple way of pooling information from the two state assignments is given. Finally, the results of the paper are summed up in a conclusion.

Needless to say that an extension of the arguments of this paper from two to any finite number of state assignments is straightforward.

2. Derivation of the first BFM condition

When one is dealing with the statistical knowledge, as is the case with density operators, then one has in mind a *random* element of the ensemble. As far as a specific individual system from the ensemble is concerned, the statistical notions like the average hardly make sense.

If P is a projector (physically: property or event), then $\mathrm{Tr}\, P\rho$ is the *probability* of possession of the property by (or of occurrence of the event on) a random system from the ensemble described by ρ. In the special case when $\mathrm{Tr}(P\rho) = 1$, i.e., when one has a *sharp* or possessed *property* (a certain event), it is easy to show that for any state decomposition $\rho = \sum_k w_k\rho_k$ $\big(\forall k,\ \ w_k > 0, \rho_k > 0, \mathrm{Tr}\,\rho_k = 1; \sum_k w_k = 1\big)$, all substates ρ_k 'inherit' the sharp property: $\forall k, \mathrm{Tr}(P\rho_k) = 1$. Analogous statements hold true for finite laboratory ensembles and subensembles that represent empirically the density operators. Hence, in terms of sharp properties, one can speak of *individual-system knowledge*, because it applies not only to a random, but also to a specific system in the ensemble.

If P and P' are two projectors (commuting or not), one must clarify in which case they can both be simultaneously sharp properties of one and the same system.

It is shown in the appendix that P is a sharp property in the state ρ if and only if

$$P \geqslant Q \quad \text{meaning} \quad PQ = Q \tag{3}$$

where Q projects onto the support of ρ. (The first relation expresses implication in the lattice of projectors and the second is its algebraic equivalent.)

Condition (3) makes it obvious that two properties P and P' can both be simultaneously sharp properties of one and the same system if and only if their greatest lower bound P_{glb} is nonzero, because then and only then do they have a common nonzero lower bound Q, which can be the support projector of a density operator. If $[P, P'] = 0$, then $P_{\text{glb}} = PP'$.

The following claim gives physical meaning to the greatest lower bound P_{glb}.

Lemma 1. *Properties P and P' are* sharp properties *in a state ρ if and only if so is their greatest lower bound P_{glb}.*

Proof. It follows immediately from the necessary and sufficient condition (3), because the two projectors have Q, the range projector of ρ, as their common lower bound if and only if $P_{\text{glb}} \geqslant Q$. □

Thus, P and P' as sharp properties can be replaced by the single sharp property P_{glb}.

Returning to the two state assignments ρ_A and ρ_B concerning one and the same system, the corresponding support projectors Q_A and Q_B are both sharp properties of the system in question as seen from (3). Then so is their greatest lower bound Q_{glb}, and it must not be zero, because zero cannot be a sharp property (cf (3)). Q_{glb} projects onto $\text{supp}(\rho_A) \cap \text{supp}(\rho_B)$. Therefore, this subspace must not be zero either. This is the first BFM condition. It is obviously valid both in finite- and infinite-dimensional state spaces.

3. The second BFM condition in an infinite-dimensional state space

An important result of Hadjisavvas [8] establishes the following claim.

Lemma 2. *A pure state $|\phi\rangle$ can appear in a state decomposition of a given density operator ρ (cf (2)) in a state space of finite or infinite dimension* if and only if

$$|\phi\rangle \in \text{ran}(\rho^{1/2}) \tag{4}$$

where $\text{ran}(\cdots)$ *denotes the range.*

If $\text{supp}(\rho)$ is finite dimensional, then $\text{ran}(\rho^{1/2}) = \text{ran}(\rho) = \text{supp}(\rho)$. But if $\text{supp}(\rho)$ is infinite dimensional, then

$$\text{ran}(\rho) \subset \text{ran}(\rho^{1/2}) \subset \text{supp}(\rho) \tag{5}$$

(proper subsets).

Let us take a simple example in which ρ_A has an infinite-dimensional range and

$$|\phi\rangle \in \left(\text{supp}(\rho_A) \ominus \text{ran}\left(\rho_A^{1/2}\right)\right) \qquad \langle\phi||\phi\rangle = 1. \tag{6a}$$

Let, further,

$$|\phi\rangle \in \text{supp}(\rho_B). \tag{6b}$$

Let, finally,

$$(\text{supp}(\rho_B) \ominus \text{span}(|\phi\rangle)) \perp (\text{supp}(\rho_A) \ominus \text{span}(|\phi\rangle)). \tag{6c}$$

Then the first BFM condition is satisfied, but $|\phi\rangle$, the only common state vector (up to a phase factor) in the supports, cannot appear in a decomposition like the first one in (2) on account of lemma 2. Therefore, the two BFM conditions are not equivalent if at least one of the state assignments has an infinite-dimensional support.

In view of lemma 2, the second BFM condition is equivalent to

$$\dim\left\{\mathrm{ran}\left(\rho_A^{1/2}\right)\cap\mathrm{ran}\left(\rho_B^{1/2}\right)\right\}\geqslant 1 \tag{7}$$

irrespectively of the dimensions of the supports. If both ranges are finite dimensional, (7) equals (1). If at least one of the ranges is infinite dimensional, the linear manifolds in (7) are proper subsets of the topologically closed subspaces appearing in (1). Then, (7) is stronger than (1), i.e., the former implies the latter, and I am not aware of any argument so far that would prove the validity of (7) as a necessary condition for the compatibility of the two state assignments.

4. How two compatible state assignments can improve each other?

We assume that (1) is valid, i.e., that the two state assignments are compatible. It may happen that $Q_{\mathrm{glb}}\equiv\mathrm{glb}(Q_A,Q_B)<Q_A$, i.e., that Q_{glb} is not a sharp property of the system in question according to ρ_A, though it is known to be if also the information from ρ_B is taken into account. In this case ρ_A contains disinformation as far as the individual system under consideration is concerned, and one may like to dispense with it. We lean on the following mathematical facts in finding a way to do so.

Lemma 3. *Let P and ρ be a projector and a density operator such that $p\equiv\mathrm{Tr}(\rho P)>0$. Then $\rho_L\equiv P\rho P/p$ is* closest *to ρ in the sense of the Hilbert–Schmidt distance in comparison with all density operators for which P is a sharp property.*

The capital L in the index is due to my liking to call ρ_L a Lüders state. The reader is familiar with it in the context of change of state in ideal measurement. To my knowledge, it was introduced by Lüders [9] (and not by von Neumann [10] as many seem to think, cf also [11]). Its above-claimed meaning concerning distance was established (in the case of the so-called non-selective measurement, when all the results of the measurement are taken into account) in previous work [12].

Proof. The proof of lemma 3 follows immediately if one takes into account the following facts: (i) all density operators are Hilbert–Schmidt ones, i.e., they are elements of the Hilbert space $\mathcal{H}_{\mathrm{HS}}$ of Hilbert–Schmidt operators. These are linear operators A such that $\mathrm{Tr}(A^\dagger A)<\infty$. (If the state space is finite dimensional, then all linear operators are Hilbert–Schmidt ones.) (ii) The superoperator $P\cdots P$ is a projector in $\mathcal{H}_{\mathrm{HS}}$. (iii) The projection of a given vector onto a given subspace of a (complex or real) unitary space has the smallest distance from the given vector in comparison with all vectors from the subspace. □

The established claim of being 'closest' may carry a mathematical elegance, but its physical meaning may be not so transparent. Therefore, we approach the Lüders state from another angle.

If ρ_A is to be changed into another density operator describing a state in which a given property P will be possessed, the statistical predictions will change in general. Still, there can be a set of predictions that should not change: those properties P' that imply P, so that when they become possessed, P remains possessed.

Lemma 4. *Let again P, ρ be given with $p \equiv \mathrm{Tr}(\rho P) > 0$. Let, further, $\mathcal{S} \equiv \{P' : P \geqslant P'\}$ be the set of all projectors implying P. Then*

$$\mathrm{Tr}(\rho P') = p[\mathrm{Tr}(\rho' P')] \tag{8}$$

for all $P' \in \mathcal{S}$ if and only if *$\rho' = \rho_L$, where ρ_L is the Lüders state (cf lemma 3).*

To my knowledge, a lemma related to the claim of lemma 4 was first proved by Bell and Nauenberg in [13]. We will resort to their argument in the proof that follows.

Proof. In the proof of lemma 4, let $\mathcal{S}'$ be the subset of $\mathcal{S}$ containing all its projectors onto one-dimensional subspaces. They can be written as $|\psi\rangle\langle\psi|$. Then one can argue as follows:

$$\forall|\psi\rangle\langle\psi| \in \mathcal{S}' : \qquad \mathrm{Tr}[\rho(|\psi\rangle\langle\psi|)] = p\{\mathrm{Tr}[\rho'(|\psi\rangle\langle\psi|)]\} \quad \Leftrightarrow \quad \langle\psi|\rho|\psi\rangle = p\langle\psi|\rho'|\psi\rangle.$$

On the other hand,

$$P|\psi\rangle = P\langle\psi||\psi\rangle|\psi\rangle = \langle\psi||\psi\rangle|\psi\rangle = |\psi\rangle.$$

Equivalently,

$$|\psi\rangle \in \mathrm{supp}(P). \tag{9}$$

Obviously, (9) is not only necessary, but also sufficient for $|\psi\rangle\langle\psi| \in \mathcal{S}'$.

Hence, the above chain of equivalences can be continued as follows:

$$\Leftrightarrow \langle\psi|(P\rho P)|\psi\rangle = p\langle\psi|\rho'|\psi\rangle.$$

Since $P\rho P$ is zero in the orthocomplement of $\mathrm{supp}(P)$, and $|\psi\rangle$ is an arbitrary state vector in this subspace, we finally have

$$\rho' = P\rho P/p$$

as claimed. □

One might still object that lemma 4 tells about statistical predictions (that should not change). We have a fixed individual system from the ensemble described by ρ_A in mind. Statistics may not be quite applicable. Let us return to the sharp properties. They do have individual-system meaning.

Lemma 5. *Let $\bar{\mathcal{S}}$ be the set of all sharp properties P' in a given state ρ and let P be a statistically possible but not necessarily sharp property of the random system in the state ρ, i.e., let $p \equiv \mathrm{Tr}(\rho P) > 0$. Let, finally, $\bar{\mathcal{S}}'$ be the subset of $\bar{\mathcal{S}}$ containing all P'* compatible *with P as observables, i.e., for which $[P', P] = 0$. Then both P and each P' from $\bar{\mathcal{S}}'$ are sharp properties in a state ρ'* if and only if *$\rho' = P\rho P/p$.*

Proof. The claim of lemma 5 is a special case of a wider claim proved in [11] as theorem 1 there. (The context was ideal measurement. But this was not relevant for the somewhat intricate proof given there.)

Returning to the two state assignments ρ_A and ρ_B and to the disinformation in the former, in view of lemmas 3, 4 and 5, the disinformation can be dispensed with or ρ_A can be improved if it is replaced by

$$\bar{\rho}_A \equiv Q_{\mathrm{glb}}\rho_A Q_{\mathrm{glb}}\big/p^A_{\mathrm{glb}} \tag{10}$$

where Q_{glb} is the greatest lower bound (in the lattice of projectors) of Q_A and Q_B, the support projectors of ρ_A and ρ_B respectively, and $p^A_{\mathrm{glb}} \equiv \mathrm{Tr}(\rho_A Q_{\mathrm{glb}})$. This probability is necessarily positive as proved in what follows. □

Lemma 6. *If ρ is a density operator with Q as its support projector, and P is another nonzero projector implying Q, then the probability* $\mathrm{Tr}(\rho P)$ *is positive.*

Proof. Let us start *ab contrario* assuming that $\mathrm{Tr}(\rho P) = 0$. We show that P is then necessarily a subprojector of or, equivalently, that it implies the null projector $(1 - Q)$ of ρ, in contradiction to the assumptions in lemma 6. To prove this, we write down a spectral form $\rho = \sum_i r_i |i\rangle\langle i|$ of ρ in terms of its positive eigenvalues and the corresponding eigenvectors, and analogously for P: $P = \sum_k |k\rangle\langle k|$. Then

$$0 = \mathrm{Tr}(\rho P) = \sum_i \sum_k r_i |\langle i||k\rangle|^2.$$

Since all terms are non-negative, all $|k\rangle$ must be orthogonal to all $|i\rangle$. □

Let us take stock of what has been achieved.

5. How much improvement has been achieved?

Naturally, the symmetric expression to (10), i.e., $\bar{\rho}_B = Q_{\mathrm{glb}}\rho_B Q_{\mathrm{glb}} \big/ p^B_{\mathrm{glb}}$ with $p^B_{\mathrm{glb}} \equiv \mathrm{Tr}(\rho_B Q_{\mathrm{glb}})$, is the improvement of ρ_B. It is desirable to clarify if there are *distinct sharp properties* in $\bar{\rho}_A$ and $\bar{\rho}_B$. Utilizing criterion (3), we take resort to the corresponding support projectors $\bar{Q}_A$ and $\bar{Q}_B$.

Theorem 1. *The improved states $\bar{\rho}_A$ and $\bar{\rho}_B$ have one and the same support projector, i.e., $\bar{Q}_A = \bar{Q}_B = Q_{\mathrm{glb}}$.*

Proof. It is obvious from (10) that $Q_{\mathrm{glb}}\bar{\rho}_A = \bar{\rho}_A$. Taking the trace, we see that Q_{glb} is a sharp property in the state $\bar{\rho}_A$. From (3) *mutatis mutandis* it follows that $Q_{\mathrm{glb}} - \bar{Q}_A$ is a projector. Evaluating $\mathrm{Tr}[(Q_{\mathrm{glb}} - \bar{Q}_A)\bar{\rho}_A]$, we obtain zero. On the other hand, due to $(Q_{\mathrm{glb}} - \bar{Q}_A) \leqslant Q_{\mathrm{glb}}$, or equivalently $(Q_{\mathrm{glb}} - \bar{Q}_A)Q_{\mathrm{glb}} = (Q_{\mathrm{glb}} - \bar{Q}_A)$, due to (10), and commutation under the trace, we, further, obtain

$$0 = \mathrm{Tr}[(Q_{\mathrm{glb}} - \bar{Q}_A)\bar{\rho}_A] = \mathrm{Tr}[(Q_{\mathrm{glb}} - \bar{Q}_A)\rho_A]. \tag{11}$$

Taking into account that $(Q_{\mathrm{glb}} - \bar{Q}_A) \leqslant Q_{\mathrm{glb}}$ and the latter is a subprojector of the support projector of ρ_A, hence also $(Q_{\mathrm{glb}} - \bar{Q}_A)$ is a subprojector of the same, we see that relation (11) and lemma 6 imply $Q_{\mathrm{glb}} - \bar{Q}_A = 0$. □

Thus, the two improved states have the same set of sharp properties. The method of sharp properties applied so far cannot lead us any further. This all generalizes trivially to the case of several state assignments.

If one of the improved state assignments turns out to be pure, then all are the same pure state. With less luck in the pooling performed so far, one could end up with mixed improved states, but still all equal ones. This would also end the necessity for further pooling. But in general the improved states can be mixed and distinct. Then further pooling is required.

A simple way of pooling information from the two state assignments is as follows. Let $0 < w < 1$. The result of pooling is obtained by averaging

$$\rho \equiv w Q_{\mathrm{glb}}\rho_A Q_{\mathrm{glb}} \big/ p^A_{\mathrm{glb}} + (1 - w) Q_{\mathrm{glb}}\rho_B Q_{\mathrm{glb}} \big/ p^B_{\mathrm{glb}}. \tag{12}$$

If the 'quantum knowledges' of both Alice and Bob are believed to be equally trustworthy, then $w = 1/2$ seems in order.

A more sophisticated way of pooling is required in some cases (see, e.g., [6, 7]).

6. Conclusion

The basic method of this paper is that of *sharp properties*. These are interpreted as being possessed by the individual quantum system about which Alice and Bob have quantum knowledges or state assignments. Using this method, the first BFM necessary condition (see the introduction) was rederived. It was shown that in the case of infinite-dimensional ranges of the density operators, the first BFM condition is valid; the second need not be. Finally, the method was utilized to improve Alice's and Bob's state assignments on account of information from each other. Thus, they end up with possibly different density operators, but they have one and the same set of sharp or possessed properties.

Appendix

We prove now that for a projector P and a statistical operator ρ with the range projector Q one has P as a *sharp property*, i.e., $\mathrm{Tr}\, P\rho = 1$, if and only if $P \geqslant Q$.

Sufficiency. The assumed relation $P \geqslant Q$ means $PQ = Q$. Further, $Q\rho = \rho$. Hence,

$$1 = \mathrm{Tr}\,\rho = \mathrm{Tr}(Q\rho) = \mathrm{Tr}(PQ\rho) = \mathrm{Tr}(P\rho).$$

Necessity. Let

$$\rho = \sum_i r_i |i\rangle\langle i| \qquad \forall i: \;\; r_i > 0$$

be a spectral form of ρ. By assumption, now one has

$$1 = \sum_i r_i \,\mathrm{Tr}(P|i\rangle\langle i|) = \sum_i r_i \langle i|P|i\rangle.$$

Subtracting this from $1 = \sum_i r_i$, one obtains

$$0 = \sum_i r_i (1 - \langle i|P|i\rangle).$$

Since always $0 \leqslant \langle i|P|i\rangle \leqslant 1$, we have

$$\forall i: \quad \langle i|P|i\rangle = 1$$

or equivalently,

$$\forall i: \quad \langle i|P^{\perp}|i\rangle = 0 \quad \Leftrightarrow \quad ||P^{\perp}|i\rangle||^2 = 0 \quad \Leftrightarrow \quad P^{\perp}|i\rangle = 0 \quad \Leftrightarrow \quad P|i\rangle = |i\rangle.$$

Since $\sum_i |i\rangle\langle i| = Q$, we finally have $PQ = Q$ as claimed.

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